

Matrix Fejér-Riesz Theorem with gaps

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Abstract

The matrix Fejér-Riesz theorem characterizes positive semidefinite matrix polynomials on the real line \mathbb{R} . We extend a characterization to arbitrary closed semialgebraic sets $K \subseteq \mathbb{R}$ by the use of matrix preorderings from real algebraic geometry. In the compact case a denominator-free characterization exists, while in the non-compact case there are counterexamples. However, there is a weaker characterization with denominators in the non-compact case. At the end we extend the results to algebraic curves.

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1. Introduction

1.1. Motivation

The matrix Fejér-Riesz theorem is the following result (For the proof see either of [8], [18], [10], [6], [4], [17], [7]).

Theorem 1.1. *Let $F(x) = \sum_{m=0}^{2N} F_m x^m$ be a $n \times n$ matrix polynomial from $M_n(\mathbb{C}[x])$ which is positive semidefinite on \mathbb{R} . Then there exists a matrix polynomial $G(x) = \sum_{m=0}^N G_m x^m \in M_n(\mathbb{C}[x])$ such that $F(x) = G(x)^* G(x)$ where $G(x)^* = \sum_{m=0}^N G_m^* x^m = \sum_{m=0}^N \overline{G_m}^T x^m = \overline{G(x)}^T$.*

In the scalar case ($n = 1$) Theorem 1.1 has already been extended to a finite union of points and intervals (not necessarily bounded) in \mathbb{R} by S. Kuhlmann and Marshall [11, Theorem 2.2]. The main problem of our paper is the following.

Problem. *Characterize univariate matrix polynomials which are positive semidefinite on a finite union of points and intervals (not necessarily bounded) in \mathbb{R} .*

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Our main results, which will be explicitly stated in Subsection 1.3, are a denominator-free generalization of Theorem 1.1 to a finite union of compact intervals in \mathbb{R} , a classification of counterexamples for a denominator-free generalization to an unbounded finite union of closed intervals in \mathbb{R} and a generalization with denominators in this case.

1.2. Notation and known results

Let $M_n(\mathbb{C}[x])$ be a set of all $n \times n$ matrix polynomials over $\mathbb{C}[x]$ equipped with the involution $F(x)^* = \overline{F(x)}^T$ where $\overline{x} = x$.

Remark 1.2. For $n = 1$ and $p(x) := \sum_{i=0}^m a_i x^i \in \mathbb{C}[x]$, the involution is $p(x)^* = \sum_{i=0}^m \overline{a_i} x^i$.

We say $F(x) \in M_n(\mathbb{C}[x])$ is *hermitian* if $F(x) = F(x)^*$. We write $\mathbb{H}_n(\mathbb{C}[x])$ for the set of all hermitian matrix polynomials from $M_n(\mathbb{C}[x])$. A matrix polynomial $F(x) \in \mathbb{H}_n(\mathbb{C}[x])$ is *positive semidefinite* in $x_0 \in \mathbb{C}$ if $v^* F(x_0) v \geq 0$ for every nonzero $v \in \mathbb{C}^n$. We denote by $\sum M_n(\mathbb{C}[x])^2$ the set of all finite sums of the expressions of the form $G(x)^* G(x)$ where $G(x) \in M_n(\mathbb{C}[x])$. We call such expressions *hermitian squares* of matrix polynomials.

The *closed semialgebraic set* associated to a finite subset $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ is given by $K_S = \{x \in \mathbb{R} : g_j(x) \geq 0, j = 1, \dots, s\}$. We define the *n-th matrix quadratic module generated by S* in $\mathbb{H}_n(\mathbb{C}[x])$ by

$$M_S^n := \left\{ \sigma_0 + \sigma_1 g_1 + \dots + \sigma_s g_s : \sigma_j \in \sum M_n(\mathbb{C}[x])^2, j = 0, \dots, s \right\},$$

and the *n-th matrix preordering generated by S* in $\mathbb{H}_n(\mathbb{C}[x])$ by

$$T_S^n := \left\{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e : \sigma_e \in \sum M_n(\mathbb{C}[x])^2 \text{ for all } e \in \{0,1\}^s \right\},$$

where $e := (e_1, \dots, e_s)$ and \underline{g}^e stands for $g_1^{e_1} \dots g_s^{e_s}$.

Remark 1.3. Note that T_S^n is the quadratic module generated by all products $\underline{g}^e, e \in \{0,1\}^s$.

We write $\text{Pos}_{\geq 0}^n(K_S)$ for the set of all $n \times n$ hermitian matrix polynomials which are positive semidefinite on K_S . We say M_S^n (resp. T_S^n) is *saturated* if $M_S^n = \text{Pos}_{\geq 0}^n(K_S)$ (resp. $T_S^n = \text{Pos}_{\geq 0}^n(K_S)$).

Theorem 1.1 can be restated in the following form.

Theorem 1.1'. *Assume the notation as above. The set $M_\emptyset^n = T_\emptyset^n$ is saturated for every $n \in \mathbb{N}$.*

The aim of this article is to study matrix generalizations of Theorem 1.1' to an arbitrary closed semialgebraic set $K \subseteq \mathbb{R}$. In this notation Problem becomes the following.

Problem’. Assume $K \subseteq \mathbb{R}$ is a closed semialgebraic set. Does there exist a finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$ and the n -th matrix quadratic module M_S^n or preordering T_S^n is saturated for every $n \in \mathbb{N}$?

Now we recall a description of a closed semialgebraic set $K \subseteq \mathbb{R}$, introduced in [11], which solves Problem’ for $n = 1$. A set $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K if it satisfies the following conditions:

- (a) If K has the least element a , then $x - a \in S$.
- (b) If K has the greatest element a , then $a - x \in S$.
- (c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.
- (d) These are the only elements of S .

Problem’ has already been solved in the following cases:

1. The preordering T_S^1 is saturated for the natural description S of K (see [11, Theorem 2.2]).
2. For $K = K_{\{x, 1-x\}} = [0, 1]$, $M_{\{x, 1-x\}}^n$ is saturated for every $n \in \mathbb{N}$ (see [5, Theorem 2.5] or [24, Theorem 7]).
3. For $K = K_{\{x\}} = [0, \infty)$, $M_{\{x\}}^n$ is saturated for every $n \in \mathbb{N}$ (see [24, Theorem 8] or [3, Proposition 3]).

Even more can be said in the case $n = 1$. There is a characterization of finite sets $S = \{g_1, \dots, g_s\} \subset \mathbb{R}[x]$ such that the preordering T_S^1 is saturated, which we now explain. We separate two possibilities according to the compactness of K_S .

1. K_S is not compact: By [11, Theorem 2.2], T_S^1 is saturated iff S contains each of the polynomials in the natural description of K_S up to scaling by positive constants.
2. K_S is compact: Write K_S as the union of pairwise disjoint points and intervals, i.e., $K_S = \cup_{j=1}^t [x_j, y_j]$ where $x_j \leq y_j$ for every $j = 1, \dots, t$. By a special case of Scheiderer’s results [22, Corollary 4.4], [21, Theorem 5.17] (which cover non-singular curves in \mathbb{R}^n), $M_S^1 = T_S^1$ and M_S^1 is saturated iff the following two conditions hold:
 - (a) For every left endpoint x_j there exists $k \in \{1, \dots, s\}$ such that $g_k(x_j) = 0$ and $g'_k(x_j) > 0$.
 - (b) For every right endpoint y_j there exists $k \in \{1, \dots, s\}$ such that $g_k(y_j) = 0$ and $g'_k(y_j) < 0$.
(For another proof see [12, Theorem 3.2].) We call every set $S \subset \mathbb{R}[x]$ which satisfies the two conditions above a *saturated description* of K_S .

Convention. An interval always has a non-empty interior.

1.3. New results

One of the main results of the paper which solves Problem’ for compact sets K is the following.

Theorem C. *Let K be a compact semialgebraic set. The n -th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ iff S is a saturated description of K (see Theorem 2.1).*

The answers to Problem' for unbounded sets K (except for a union of one or two unbounded intervals and a point) are given by the following result.

Theorem D. *Let K be an unbounded closed semialgebraic set.*

The n -th matrix quadratic module M_S^n is saturated for the natural description S of K and every $n \in \mathbb{N}$ if K is either of the following:

1. *An unbounded interval (by Theorem 1.1' and [24, Theorem 8]).*
2. *A union of two unbounded intervals (see Proposition 3.1).*

The n -th matrix preordering T_S^n is not saturated for any finite set $S \subset \mathbb{R}[x]$ such that $K = K_S$ in the following cases (see Theorem 3.2):

1. *$n \geq 2$ and K contains at least two intervals with at least one of them bounded.*
2. *$n \geq 2$ and K is a union of an unbounded interval and m isolated points with $m \geq 2$.*
3. *$n \geq 2$ and K is a union of two unbounded intervals and m isolated points with $m \geq 2$.*

In the remaining cases of a union of one or two unbounded intervals and a point not covered by Theorems C and D we state the following conjecture based on the investigation of some examples.

Conjecture. *Let $K \subseteq \mathbb{R}$ be either of the following:*

1. *A union of an unbounded interval and a point.*
2. *A union of two unbounded intervals and a point.*

Suppose S is the natural description of K . Then the n -th matrix preordering T_S^n is saturated for every natural number $n > 1$.

Note that by an appropriate substitution of variables both cases covered by Conjecture are equivalent.

For the unbounded sets K with a negative answer to Problem' we obtain the following characterization of the set $\text{Pos}_{\leq 0}^n(K)$.

Theorem E. *Let K be an unbounded closed semialgebraic set with a natural description S and $n \in \mathbb{N}$. Then the following statements are equivalent:*

1. *$F \in \text{Pos}_{\leq 0}^n(K)$.*
2. *For every $w \in \mathbb{C}$ there exists $h \in \mathbb{R}[x]$ such that $h(w) \neq 0$ and $h^2 F \in T_S^n$ (see Theorem 3.5).*
3. *For every $w \in \mathbb{C} \setminus K$ there exists $k_w \in \mathbb{N} \cup \{0\}$ such that*

$$((x - \overline{w})(x - w))^{k_w} F \in T_S^n$$

(see Corollary 4.3 and Remark 4.4).

4. $(1 + x^2)^k F \in T_S^n$ for some $k \in \mathbb{N} \cup \{0\}$ (Take $w = i$ in 3.).

The following table summarizes [11, Theorem 2.2], Theorems C, D and Conjecture.

| K | A | B |
|--|-----|-----|
| a bounded set | Yes | Yes |
| an unbounded interval | Yes | Yes |
| a union of an unbounded interval and an isolated point | Yes | C |
| a union of an unbounded interval and m isolated points with $m \geq 2$ | Yes | No |
| a union of two unbounded intervals | Yes | Yes |
| a union of two unbounded intervals and an isolated point | Yes | C |
| a union of two unbounded intervals and m isolated points with $m \geq 2$ | Yes | No |
| includes a bounded and an unbounded interval | Yes | No |

- A := The preordering T_S^1 is saturated for the natural description S of K .
- B := The n -th matrix preordering T_S^n is saturated for the natural description S of K and every integer $n \in \mathbb{N}$.
- C := See Conjecture.

Remark 1.4. 1. Since T_S^1 is saturated for the natural description S of K , it follows that if T_S^n is not saturated for some $n \in \mathbb{N}$, then $T_{S_1}^n$ is not saturated for any finite set S_1 satisfying $K_{S_1} = K$.

2. The classification covers all closed semialgebraic sets $K \subseteq \mathbb{R}$. A set K is *regular* if it is equal to the closure of its interior. For regular sets $K \subseteq \mathbb{R}$ the classification is complete.

2. Saturated descriptions of a compact set $K \subset \mathbb{R}$ generate saturated n -th matrix quadratic modules

The solution to Problem' from the Introduction for a compact set K is the main result of this section (see Theorem 2.1 below). It also characterizes all finite sets S such that the quadratic module M_S^n is saturated for every natural number $n \in \mathbb{N}$.

Theorem 2.1. *Suppose K is a non-empty compact semialgebraic set in \mathbb{R} . The n -th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ iff S a saturated description of K .*

The main ingredients in the proof of Theorem 2.1 are:

1. The $n = 1$ case [21, Theorem 5.17].

2. The “ h^2F -proposition” (See Proposition 2.2 below. The proof uses the idea of diagonalizing matrix polynomials from [23, 4.3].).
3. Getting rid of h^2 in “ h^2F -proposition” (The proof uses [20, Proposition 2.7], which is Proposition 2.6 below.).

2.1. “ h^2F -proposition”

We call the following result “ h^2F -proposition”.

Proposition 2.2. *Suppose K is a non-empty compact semialgebraic set in \mathbb{R} with a saturated description S . Then, for any $F \in \mathbb{H}_n(\mathbb{C}[x])$ such that $F \succeq 0$ on K and every point $x_0 \in \mathbb{C}$, there exists $h \in \mathbb{R}[x]$ such that $h(x_0) \neq 0$ and $h^2F \in M_S^n$.*

To prove Proposition 2.2 we need Lemmas 2.3 and 2.4 below.

Lemma 2.3. *Let $G = [g_{kl}]_{kl} \in M_n(\mathbb{C}[x])$. For every $1 \leq k \leq l \leq n$ there exist unitary matrices $U_{kl} \in M_n(\mathbb{R})$ and $V_{kl} \in M_n(\mathbb{C})$ such that*

$$U_{kl}GU_{kl}^* = \begin{bmatrix} p_{kl} & * \\ * & * \end{bmatrix}, \quad V_{kl}GV_{kl}^* = \begin{bmatrix} r_{kl} & * \\ * & * \end{bmatrix},$$

where

$$\begin{aligned} p_{kl} &= \begin{cases} g_{kl}, & \text{for } 1 \leq k = l \leq n \\ \frac{1}{2}(g_{kl} + g_{lk} + g_{kk} + g_{ll}), & \text{for } 1 \leq k < l \leq n \end{cases}, \\ r_{kl} &= \begin{cases} g_{kl}, & \text{for } 1 \leq k = l \leq n \\ \frac{i}{2}(-g_{kl} + g_{lk}) + \frac{1}{2}(g_{kk} + g_{ll}), & \text{for } 1 \leq k < l \leq n \end{cases}. \end{aligned}$$

Proof. We define $U_{11} = V_{11} := I_n$, $U_{kk} = V_{kk} := P_k$ for $k = 2, \dots, n$, where P_k denotes the permutation matrix which permutes the first row and the k -th row.

For $1 \leq k < l \leq n$, define $U_{kl} := P_k S_{kl}$ where $S_{kl} = \left(s_{pr}^{(kl)} \right)_{pr} \in M_n(\mathbb{R})$ is the matrix with $s_{kk}^{(kl)} = s_{kl}^{(kl)} = s_{lk}^{(kl)} = \frac{1}{\sqrt{2}}$, $s_{ll}^{(kl)} = -\frac{1}{\sqrt{2}}$, $s_{pp}^{(kl)} = 1$ if $p \notin \{k, l\}$ and $s_{pr}^{(kl)} = 0$ otherwise.

For $1 \leq k < l \leq n$, define $V_{kl} := P_k \tilde{S}_{kl}$ where $\tilde{S}_{kl} = \left(\tilde{s}_{pr}^{(kl)} \right)_{pr} \in M_n(\mathbb{C})$ is the matrix with $\tilde{s}_{kk}^{(kl)} = \tilde{s}_{lk}^{(kl)} = \frac{1}{\sqrt{2}}$, $\tilde{s}_{kl}^{(kl)} = \frac{i}{\sqrt{2}}$, $\tilde{s}_{ll}^{(kl)} = -\frac{i}{\sqrt{2}}$, $\tilde{s}_{pp}^{(kl)} = 1$ if $p \notin \{k, l\}$ and $\tilde{s}_{pr}^{(kl)} = 0$ otherwise. \square

Lemma 2.4. *For $F = \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \in \mathbb{H}_n(\mathbb{C}[x])$ where $a = a^* \in \mathbb{R}[x]$, $\beta \in M_{1,n-1}(\mathbb{C}[x])$ and $C \in \mathbb{H}_{n-1}(\mathbb{C}[x])$ it holds that*

$$\begin{aligned} (i) \quad a^4 \cdot F &= \begin{bmatrix} a^* & 0 \\ \beta^* & a^* I_{n-1} \end{bmatrix} \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^* \beta) \end{bmatrix} \begin{bmatrix} a & \beta \\ 0 & a I_{n-1} \end{bmatrix}. \\ (ii) \quad \begin{bmatrix} a^3 & 0 \\ 0 & a(aC - \beta^* \beta) \end{bmatrix} &= \begin{bmatrix} a^* & 0 \\ -\beta^* & a^* I_{n-1} \end{bmatrix} \cdot F \cdot \begin{bmatrix} a & -\beta \\ 0 & a I_{n-1} \end{bmatrix}. \end{aligned}$$

Proof. Easy computation. \square

Proof of Proposition 2.2. The proof is by induction on the size n of the matrix polynomials. For $n = 1$ the proposition holds by the scalar case (We take $h = 1$ and use [21, Theorem 5.17] and [22, Corollary 4.4].). Suppose the proposition holds for $n - 1$. We will prove that it holds for n . Let us take $F := [f_{kl}]_{kl} \in \mathbb{H}_n(\mathbb{C}[x])$ where $F \succeq 0$ on K . Let us define

$$c(x) := \begin{cases} x - x_0, & x_0 \in \mathbb{R} \\ (x - x_0)(x - \overline{x_0}), & x_0 \in \mathbb{C} \setminus \mathbb{R} \end{cases}.$$

If $F \equiv 0$, we can take $h = 1$. Otherwise $F \not\equiv 0$ and we write

$$F = c^m G,$$

where $m \in \mathbb{N} \cup \{0\}$, $G = [g_{kl}]_{kl} \in \mathbb{H}_n(\mathbb{C}[x])$ and

$$G(x_0) = [g_{kl}(x_0)]_{kl} \neq 0. \quad (1)$$

Claim. One of the following two cases applies:

Case 1: $g_{k_0 k_0}(x_0) \neq 0$ for some $k_0 \in \{1, \dots, n\}$.

Case 2: $g_{kk}(x_0) = 0$ for all $k \in \{1, \dots, n\}$ and for some $1 \leq k_0 < l_0 \leq n$ we have

$$\Re(g_{k_0 l_0})(x_0) \neq 0 \quad \text{or} \quad \Im(g_{k_0 l_0})(x_0) \neq 0,$$

where $\Re(g_{k_0 l_0}) := \frac{g_{k_0 l_0} + \overline{g_{k_0 l_0}}}{2} \in \mathbb{R}[x]$ and $\Im(g_{k_0 l_0}) := \frac{g_{k_0 l_0} - \overline{g_{k_0 l_0}}}{2i} \in \mathbb{R}[x]$.

Proof of Claim. Let us assume that none of the two cases applies. Then $\Re(g_{kl})(x_0) = \Im(g_{kl})(x_0) = 0$ for all $1 \leq k \leq l \leq n$. Let us take $l < k$. Since $G \in \mathbb{H}_n(\mathbb{C}[x])$ is hermitian, it follows that $g_{lk} = \overline{g_{kl}} = \Re g_{kl} - i \cdot \Im g_{kl}$. Therefore $g_{lk}(x_0) = \Re g_{kl}(x_0) - i \cdot \Im g_{kl}(x_0) = 0$. Hence $g_{kl}(x_0) = 0$ for all $k, l \in \{1, \dots, n\}$. This is a contradiction with (1) and proves Claim.

Let U_{kl} , V_{kl} , p_{kl} , r_{kl} be as in Lemma 2.3. We study each case from Claim separately:

Case 1: We define $T_{k_0 k_0} := U_{k_0 k_0}$, $\tilde{g}_{k_0 k_0} := g_{k_0 k_0}$. Notice that $\tilde{g}_{k_0 k_0}(x_0) = g_{k_0 k_0}(x_0) \neq 0$.

Case 2: We will separate three subcases:

Subcase 2.1. $p_{k_0 l_0}(x_0) \neq 0$: We define $T_{k_0 l_0} := U_{k_0 l_0}$, $\tilde{g}_{k_0 l_0} := p_{k_0 l_0}$. Notice that $\tilde{g}_{k_0 l_0}(x_0) \neq 0$.

Subcase 2.2. $r_{k_0 l_0}(x_0) \neq 0$: We define $T_{k_0 l_0} := V_{k_0 l_0}$, $\tilde{g}_{k_0 l_0} := r_{k_0 l_0}$. Notice that $\tilde{g}_{k_0 l_0}(x_0) \neq 0$.

Subcase 2.3. $p_{k_0 l_0}(x_0) = r_{k_0 l_0}(x_0) = 0$: We will prove that this subcase does not happen. By definition and assumptions we have

$$\begin{aligned} p_{k_0 l_0}(x_0) &= \frac{1}{2}(g_{k_0 l_0} + g_{l_0 k_0} + g_{k_0 k_0} + g_{l_0 l_0})(x_0) = \frac{1}{2}(g_{k_0 l_0} + g_{l_0 k_0})(x_0) = \\ &= (\Re g_{k_0 l_0})(x_0) \\ r_{k_0 l_0}(x_0) &= \frac{i}{2}(-g_{k_0 l_0} + g_{l_0 k_0})(x_0) + \frac{1}{2}(g_{k_0 k_0} + g_{l_0 l_0})(x_0) = \\ &= \frac{i}{2}(-g_{k_0 l_0} + g_{l_0 k_0})(x_0) = (\Im g_{k_0 l_0})(x_0) \end{aligned}$$

Since we are in Case 2, $(\Re g_{k_0 l_0})(x_0) \neq 0$ or $(\Im g_{k_0 l_0})(x_0) \neq 0$. Contradiction. Hence Subcase 2.3 never happens.

To avoid repetition in what follows we define $k_0 = l_0$ if we are in Case 1. If we write $T_{k_0 l_0} G T_{k_0 l_0}^* = \begin{bmatrix} \tilde{g}_{k_0 l_0} & \tilde{\beta} \\ \tilde{\beta}^* & \tilde{C} \end{bmatrix}$ with $\tilde{\beta} \in M_{1, n-1}(\mathbb{C}[x])$ and $\tilde{C} \in M_{n-1}(\mathbb{C}[x])$, then $T_{k_0 l_0} F T_{k_0 l_0}^* = \begin{bmatrix} c^m \tilde{g}_{k_0 l_0} & c^m \tilde{\beta} \\ (c^m \tilde{\beta})^* & c^m \tilde{C} \end{bmatrix} =: \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix}$. Therefore by part (i) of Lemma 2.4 and dividing by c^{4m} , it follows that

$$\tilde{g}^2 F = T_{k_0 l_0}^* \begin{bmatrix} \tilde{g}_{k_0 l_0}^* & 0 \\ \tilde{\beta}^* & \tilde{g}_{k_0 l_0} I_{n-1} \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \tilde{g}_{k_0 l_0} & \tilde{\beta} \\ 0 & \tilde{g}_{k_0 l_0} I_{n-1} \end{bmatrix} T_{k_0 l_0},$$

where

$$\begin{aligned} \tilde{g} &= \tilde{g}_{k_0 l_0}^2 \in \mathbb{H}_1(\mathbb{C}[x]) = \mathbb{R}[x] \\ d &= c^m \tilde{g}_{k_0 l_0}^3 \in \mathbb{H}_1(\mathbb{C}[x]) = \mathbb{R}[x], \\ D &= c^m \tilde{g}_{k_0 l_0} (\tilde{g}_{k_0 l_0} \tilde{C} - \tilde{\beta}^* \tilde{\beta}) \in \mathbb{H}_{n-1}(\mathbb{C}[x]). \end{aligned}$$

By part (ii) of Lemma 2.4 and dividing by c^{2m} , we have also

$$\begin{bmatrix} d & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} \tilde{g}_{k_0 l_0}^* & 0 \\ -\tilde{\beta}^* & \tilde{g}_{k_0 l_0} I_{n-1} \end{bmatrix} T_{k_0 l_0} F T_{k_0 l_0}^* \begin{bmatrix} \tilde{g}_{k_0 l_0} & -\tilde{\beta} \\ 0 & \tilde{g}_{k_0 l_0} I_{n-1} \end{bmatrix}$$

It follows that $d \geq 0$, $D \succeq 0$ on K . By the induction hypothesis used for the polynomial $D \in \mathbb{H}_{n-1}(\mathbb{C}[x])$, there exists $h_1 \in \mathbb{R}[x]$ such that $h_1(x_0) \neq 0$ and $h_1^2 D \in M_S^{n-1}$. By the scalar case [21, Theorem 5.17] and [22, Corollary 4.4], $h_1^2 d \in M_S^1$. Hence $h^2 F \in M_S^n$ where $h = h_1 \tilde{g} \in \mathbb{R}[x]$ and $h(x_0) \neq 0$. This concludes the proof. \square

Remark 2.5. By keeping track on the degree of h and using [12, Theorem 4.1], we can prove more in Proposion 2.2 above. Namely, h can be chosen of degree at most $\deg(F)(3^n - 1)$ and if $S = \{g_1, \dots, g_s\}$ is the natural description of K , then $F = \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \in T_S^n$ for some $\sigma_e \in M_n(\mathbb{C}[x])^2$ with $\deg(\sigma_e \underline{g}^e) \leq \deg(h^2 F)$.

2.2. Getting rid of h^2 in “ h^2F -proposition”

To get rid of h^2 in “ h^2F -proposition”, which proves Theorem 2.1, we will use [20, Proposition 2.7]:

Proposition 2.6. *Suppose R is a commutative ring with 1 and $\mathbb{Q} \subseteq R$. Let $\Phi : R \rightarrow C(K, \mathbb{R})$ be a ring homomorphism, where K is a topological space which is compact and Hausdorff. Suppose $\Phi(R)$ separates points in K . Suppose $f_1, \dots, f_k \in R$ are such that $\Phi(f_j) \geq 0$, $j = 1, \dots, k$ and $(f_1, \dots, f_k) = (1)$. Then there exist $s_1, \dots, s_k \in R$ such that $s_1 f_1 + \dots + s_k f_k = 1$ and such that each $\Phi(s_j)$ is strictly positive.*

Proof of Theorem 2.1. By [22, Corollary 4.4] and [21, Theorem 5.17], M_S^1 is saturated if and only if S is a saturated description of K . Therefore we have to prove only the if part. Let S be a saturated description of K . We will prove that M_S^n is saturated for every $n \in \mathbb{N}$. Let $R := \mathbb{R}[x]$ and $\Phi : R \rightarrow C(K, \mathbb{R})$ be the natural map, i.e., $\Phi(f) = f|_K$. Take $F \in \text{Pos}_{\geq 0}^n(K)$. We will prove that $F \in M_S^n$. Let $I := \langle h^2 \in \mathbb{R}[x] : h^2 F \in M_S^n \rangle$ be the ideal in $\mathbb{R}[x]$ generated by all h^2 where $h \in \mathbb{R}[x]$ is such that $h^2 F \in M_S^n$. Since $\mathbb{R}[x]$ is a principal ideal domain, there exists a polynomial $p \in \mathbb{R}[x]$ such that $I = \langle p \rangle$. If I was a proper ideal, all its elements would have a common zero $x_0 \in \mathbb{C}$. By Proposition 2.2, there exists $h \in \mathbb{R}[x]$ such that $h(x_0) \neq 0$ and $h^2 F \in M_S^n$. Since h belongs to I , it follows that I is not a proper ideal and hence $I = \mathbb{R}[x]$. By Proposition 2.6, there exist $s_1, \dots, s_k \in \text{Pos}_{\geq 0}^1(K)$ and $h_1, \dots, h_k \in I$ such that $\sum_{j=1}^k s_j h_j^2 = 1$. Hence $\sum_{j=1}^k s_j h_j^2 F = F \in M_S^n$, which concludes the proof. \square

Remark 2.7. 1. There is another proof of Theorem 2.1 which uses Proposition 2.2 just for the boundary points of K . We outline the main idea. There exists $h \in \mathbb{R}[x]$ such that $h \in \text{Pos}_{\geq 0}^1(\mathbb{R})$, $h(x_0) > 0$ for every boundary point of K and $hF \in M_S^n$ (Take $h = \sum_{x_0 \in \partial K} h_{x_0}^2$ where ∂K is the boundary of K and h_{x_0} is the polynomial from Proposition 2.2 for the point x_0). Now multiply every member of the set S by h to obtain the set S_1 which satisfies conditions of [21, Corollary 5.17]. Thus $M_S^1 = M_{S_1}^1$ and $hF \in M_{S_1}^n$. This means there exist $\sigma_j \in \sum M_n(\mathbb{C}[x])^2$ such that $hF = \sigma_0 + \sigma_1 h g_1 + \dots + \sigma_s h g_s$. From here it is easy to see that $F = \tau_0 + \sigma_1 g_1 + \dots + \sigma_s g_s$ for some $\tau_0 \in \sum M_n(\mathbb{C}[x])^2$ and hence $F \in M_S^n$. 2. By Remark 2.5, the degree of h in Proposition 2.2 and the degrees of summands in the expression of $h^2 F$ as the element of the preordering T_S^n generated by the natural description S of K can be bounded by the degree of F and n . It would be interesting to know if the same holds for F and an arbitrary compact set K . It can be shown this is true for a finite set K . The degrees can be bounded by $\max(\deg(F), |K| - 1)$.

3. Unbounded sets K without saturated T_S^2 for any finite sets S with $K_S = K$

The answer to the question of Problem’ for unbounded sets K is positive for an unbounded interval by Theorem 1.1’ (if $K = \mathbb{R}$) and [24, Theorem 8]

(if $K = [a, \infty)$). It is also easy to derive a positive answer for a union of two unbounded intervals from the case $K = [a, b]$:

Proposition 3.1. *Let $K = (-\infty, a] \cup [b, \infty)$ be a union of two unbounded intervals where $a, b \in \mathbb{R}$ and $a < b$. Then the quadratic module $M_{\{(x-a)(x-b)\}}^n$ is saturated for every $n \in \mathbb{N}$.*

Proof. By a linear change of variables, we may assume that $K = (-\infty, -1] \cup [1, \infty)$. Note that $F \in \text{Pos}_{\geq 0}^n(K)$ is of even degree. We define

$$F_1(x) = x^{\deg(F)} F\left(\frac{1}{x}\right)$$

and observe that $F_1 \succeq 0$ on $[-1, 1]$. By [5, Theorem 2.5] and by the identity

$$1 \pm x = \frac{(1 \pm x)^2 + (x+1)(1-x)}{2},$$

there exist matrix polynomials G_1, H_1 such that

$$F_1(x) = G_1(x)^* G_1(x) + H_1(x)^* H_1(x)(x+1)(1-x),$$

$$\deg(G_1) \leq \left\lfloor \frac{\deg(F_1)}{2} \right\rfloor \leq \frac{\deg(F)}{2},$$

$$\deg(H_1) \leq \left\lfloor \frac{\deg(F_1) - 1}{2} \right\rfloor \leq \left\lfloor \frac{\deg(F) - 1}{2} \right\rfloor = \frac{\deg(F)}{2} - 1.$$

Therefore

$$\begin{aligned} F(x) &= x^{\deg(F)} F_1\left(\frac{1}{x}\right) \\ &= x^{\deg(F)} \left(G_1\left(\frac{1}{x}\right)^* G_1\left(\frac{1}{x}\right) + H_1\left(\frac{1}{x}\right)^* H_1\left(\frac{1}{x}\right) \left(\frac{1}{x} + 1\right) \left(1 - \frac{1}{x}\right) \right) \\ &=: G(x)^* G(x) + H(x)^* H(x) (1+x)(x-1), \end{aligned}$$

where

$$G(x) := x^{\frac{\deg(F)}{2}} G_1\left(\frac{1}{x}\right), \quad H := x^{\frac{\deg(F)}{2}-1} H_1\left(\frac{1}{x}\right)$$

are matrix polynomials. \square

The negative answer to the question of Problem' for almost all remaining unbounded sets K (except for a union of an unbounded interval and a point or a union of two unbounded intervals and a point) and all $n \geq 2$ is the main result of this section.

Theorem 3.2. *Let an unbounded closed semialgebraic set $K \subseteq \mathbb{R}$ satisfy either of the following:*

1. K contains at least two intervals with at least one of them bounded.
2. K is a union of an unbounded interval and m isolated points with $m \geq 2$.

3. K is a union of two unbounded intervals and m isolated points with $m \geq 2$.
 If $S \subset \mathbb{R}[x]$ is a finite set with $K_S = K$, then the 2-nd matrix preordering T_S^2 is not saturated.

It is sufficient to prove Theorem 3.2 for the natural description S of K by the following lemma.

Lemma 3.3. *Let $K \subseteq \mathbb{R}$ be an unbounded closed semialgebraic set with the natural description S . Let $S_1 \subset \mathbb{R}[x]$ be a finite set such that $K_{S_1} = K$. For every $n \in \mathbb{N}$ such that the n -th matrix preordering T_S^n is not saturated, also the n -th matrix preordering $T_{S_1}^n$ is not saturated.*

Proof. Let us write $S := \{g_1, \dots, g_s\}$ and $S_1 := \{f_1, \dots, f_t\}$. We have to show that every matrix polynomial F from $T_{S_1}^n$ also belongs to T_S^n . A matrix polynomial F from $T_{S_1}^n$ is of the form

$$F = \sum_{e' \in \{0,1\}^t} \tau_{e'} f_1^{e'_1} \dots f_t^{e'_t}, \quad (2)$$

where $e' := (e'_1, \dots, e'_t)$ and $\tau_{e'} \in \sum M_n(\mathbb{C}[x])^2$. By [11, Theorem 2.2], the preordering T_S^1 is saturated and thus for each j there exist $\sigma_{e,j} \in \sum \mathbb{R}[x]^2$ such that

$$f_j = \sum_{e \in \{0,1\}^s} \sigma_{e,j} g_1^{e_1} \dots g_s^{e_s}, \quad (3)$$

where $e := (e_1, \dots, e_s)$. Plugging (3) into (2) and rearranging terms we obtain $F \in T_S^n$. This concludes the proof. \square

In the remaining part of this section we will prove Theorem 3.2. The major step will be Proposition 3.4.

Let K be a closed semialgebraic set with a natural description $S = \{g_1, \dots, g_s\}$. For $n \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{0\}$ we define the set

$$T_{S,d}^n := \left\{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e : \sigma_e \in \sum M_n(\mathbb{C}[x])^2 \text{ and } \deg(\sigma_e \underline{g}^e) \leq d \ \forall e \in \{0,1\}^s \right\}.$$

Proposition 3.4. *Let $K = [x_1, x_2] \cup [x_3, \infty)$ be a union of a bounded and an unbounded interval where $x_1 < x_2 < x_3$. Let us define the polynomial*

$$F_k(x) := \begin{bmatrix} x + A(k) & D(k) \\ D(k) & x^2 + B(k)x + C(k) \end{bmatrix},$$

where

$$\begin{aligned} A(k) &:= k - x_1, \\ B(k) &:= -k - x_2 - x_3, \\ C(k) &:= k^2 + k(-x_1 + x_2 + x_3) + x_2 x_3, \\ D(k) &:= \sqrt{A(k)C(k) + x_1 x_2 x_3} = \\ &= \sqrt{k^3 + k^2(-2x_1 + x_2 + x_3) + k(x_2 x_3 + x_1^2 - x_1 x_2 - x_1 x_3)}. \end{aligned}$$

We define $p_k(x) := x^2 + B(k)x + C(k)$. For every $k \in \mathbb{R}$ which satisfies

$$k > 0, \quad (4)$$

$$D(k)^2 = k^3 + k^2(-2x_1 + x_2 + x_3) + k(x_2x_3 + x_1^2 - x_1x_2 - x_1x_3) > 0, \quad (5)$$

$$p_k\left(-\frac{B(k)}{2}\right) = \frac{3}{4}k^2 + k\left(-x_1 + \frac{x_2 + x_3}{2}\right) - \left(\frac{x_2 - x_3}{2}\right)^2 > 0, \quad (6)$$

the matrix polynomials $F_k(x)$ belongs to $\text{Pos}_{\geq 0}^2(K)$, but:

Claim 1. $F_k \notin T_{S_1}^2$ where S_1 is the natural description of any set K_1 of the form

$$[x_1, x_2] \cup \bigcup_{j=1}^m [x_{2j+1}, x_{2j+2}] \cup [x_{2m+3}, \infty) \subseteq K$$

with $m \in \mathbb{N} \cup \{0\}$ and $x_j \leq x_{j+1}$ for each j (and $x_1 < x_2 < x_3$). In particular,

$$F_k(x) \notin T_S^2,$$

where S is the natural description of K .

Claim 2. $F_k \notin T_{S_{2,2}}^2$ where S_2 is the natural description of any set K_2 of the form

$$[x_1, x_2] \cup \bigcup_{j=3}^m \{x_j\} \subset K$$

with $m \in \mathbb{N}$, $m \geq 4$ and $x_j < x_{j+1}$ for each j .

Proof. First we will prove that $F_k(x)$ belongs to $\text{Pos}_{\geq 0}^2(K)$ for every $k \in \mathbb{R}$ satisfying the conditions (4)-(6). Note that every sufficiently large k satisfies the conditions (4)-(6). Condition (5) ensures that $D(k) \in \mathbb{R}$ and hence $F \in \mathbb{H}_n(\mathbb{R}[x])$. The determinant of $F_k(x)$ is $(x - x_1)(x - x_2)(x - x_3) \in \text{Pos}_{\geq 0}^1(K)$. The upper left corner of F is non-negative for $x \geq x_1 - k$ and hence it belongs to $\text{Pos}_{\geq 0}^1(K)$ by (4). The lower right corner is a quadratic polynomial $p_k(x)$ with a vertex in $x = \frac{-B(k)}{2}$. Since k satisfies (6), $p_k\left(\frac{-B(k)}{2}\right) > 0$. So $p_k(x)$ is positive on \mathbb{R} and hence $p_k \in \text{Pos}_{\geq 0}^1(K)$. Since all principal minors of $F_k(x)$ are non-negative on K , the conclusion $F_k(x) \in \text{Pos}_{\geq 0}^2(K)$ follows.

We will separately prove both claims of the theorem.

Proof of Claim 1. The set

$$\underbrace{\{x - x_1\}}_{g_1(x)}, \underbrace{\{(x - x_2)(x - x_3)\}}_{g_2(x)}, \dots, \underbrace{\{(x - x_{2m+2})(x - x_{2m+3})\}}_{g_{m+2}(x)}$$

is the natural description S_1 of K_1 . We will prove that $F_k(x) \notin T_{S_1}^2$ by contradiction. Let us assume $F_k \in T_{S_1}^2$. Then for every $e := (e_1, \dots, e_{m+2}) \in \{0, 1\}^{m+2}$ there exists $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$, such that

$$F_k = \sum_{e \in \{0,1\}^{m+2}} \sigma_e g_1^{e_1} \cdots g_{m+2}^{e_{m+2}}. \quad (7)$$

By the degree comparison of both sides of (7), there exist $\sigma_j \in \sum M_n(\mathbb{C}[x])^2$, such that

$$F_k(x) = \sigma_0 + \sigma_1(x - x_1) + \sum_{j=1}^{m+1} \sigma_{j+1}(x - x_{2j})(x - x_{2j+1}), \quad (8)$$

$$\deg(\sigma_0) \leq 2, \quad \deg(\sigma_j) = 0 \text{ for } j = 1, \dots, m+2.$$

By observing the monomial x^2 on both sides of (8), it follows that $\sigma_2 = \begin{bmatrix} 0 & 0 \\ 0 & k_0 \end{bmatrix}$ for some $k_0 \in [0, 1]$. Equivalently, (8) can be written as

$$F_k(x) - \sigma_2(x - x_2)(x - x_3) = \sigma_0 + \sigma_1(x - x_1) + \sum_{j=2}^{m+1} \sigma_{j+1}(x - x_{2j})(x - x_{2j+1}).$$

The right-hand side belongs to $\text{Pos}_{\geq 0}^2(\hat{K}_1)$ where $\hat{K}_1 = K_1 \cup [x_2, x_3]$. We will prove that the left-hand side does not belong to $\text{Pos}_{\geq 0}^2(\hat{K}_1)$, which is a contradiction. The determinant of the left-hand side is

$$q(x) := (x - x_2)(x - x_3)(x(1 - k_0) - (x_1 - x_1k_0 + kk_0)).$$

There are two cases to consider: $k_0 = 0$ and $k_0 > 0$. In the first case, $q(x) = (x - x_1)(x - x_2)(x - x_3)$ which is negative on (x_2, x_3) , a contradiction with $q|_{\hat{K}_1} \geq 0$. In the second case, $q(x_1) = (x_1 - x_2)(x_1 - x_3)(-kk_0) < 0$, which is also a contradiction with $q|_{\hat{K}_1} \geq 0$. Thus

$$F_k(x) - \sigma_2(x - x_2)(x - x_3) \notin \text{Pos}_{\geq 0}^2(\hat{K}_1),$$

which is a contradiction. Therefore F_k cannot be expressed in the form (7) and so $F_k \notin T_{S_1}^2$.

Proof of Claim 2. The set

$$\underbrace{\{x - x_1\}}_{g_1(x)}, \underbrace{\{(x - x_2)(x - x_3)\}}_{g_2(x)}, \dots, \underbrace{\{(x - x_{m-1})(x - x_m)\}}_{g_{m-1}(x)}, \underbrace{\{x_m - x\}}_{g_m(x)}$$

is the natural description S_2 of K_2 . If $F_k \in T_{S_2, 2}^2$, then there exist $\tau_j \in \sum M_n(\mathbb{C}[x])^2$ such that

$$F_k(x) = \tau_0 + \tau_1(x - x_1) + \sum_{j=2}^{m-1} \tau_j(x - x_j)(x - x_{j+1}) + \tau_m(x_m - x) + \tau_{m+1}(x - x_1)(x_m - x), \quad (9)$$

$$\deg(\tau_0) \leq 2, \quad \deg(\tau_j) = 0 \text{ for } j = 1, \dots, m+1.$$

From (9) it follows that

$$(F_k(x) - \tau_j(x - x_j)(x - x_{j+1}))|_{K_2} \succeq 0 \quad \text{for } j = 2, \dots, m-1. \quad (10)$$

From (10) it follows that

$$\ker F_k(x_1) \subseteq \ker \tau_j, \ker F_k(x_2) \subseteq \ker \tau_j \quad \text{for } j = 3, \dots, m-1.$$

Since $\ker F_k(x_1) \oplus \ker F_k(x_2) = \mathbb{C}^2$, we conclude that $\tau_j = 0$ for $j = 3, \dots, m-1$.

Hence (9) becomes

$$F_k(x) = \tau_0 + \tau_1(x-x_1) + \tau_2(x-x_2)(x-x_3) + \tau_m(x_m-x) + \tau_{m+1}(x-x_1)(x_m-x),$$

or equivalently,

$$F_k(x) - \tau_2(x-x_2)(x-x_3) = \tau_0 + \tau_1(x-x_1) + \tau_m(x_m-x) + \tau_{m+1}(x-x_1)(x_m-x). \quad (11)$$

Since the determinant of the left hand side is of degree 4 and is divisible by $(x-x_1)(x-x_2)(x-x_3)$ (divisibility by $x-x_1$ is due to $\ker F_k(x_1) \neq \{0\}$ and (10) for $j=2$), it cannot be non-negative on $[x_1, x_m]$ (This follows by a simple geometric argument.). Hence the left-hand side of (11) does not belong to $\text{Pos}_{\geq 0}^2([x_1, x_m])$, while the right-hand side does. This is a contradiction and thus $F_k \notin T_{S_2, 2}^2$. \square

Proof of Theorem 3.2.1. By Lemma 3.3, we may assume that S is the natural description of K . Let us write K in the form $K_0 \cup K_1$ where K_0 is the set of isolated points of K and K_1 is the regular part of K (i.e., does not have isolated points). We separate three cases depending on the form of K_1 .

Case 1: K_1 is bounded from below and unbounded from above. Let us divide the isolated part K_0 into disjoint sets K_{01}, K_{02} where in K_{01} are all those points which are smaller than the minimum of K_1 and in K_{02} all the others. The set $K_2 := K_1 \cup K_{02}$ is of the form

$$[x_1, x_2] \cup \bigcup_{j=1}^p [x_{2j+1}, x_{2j+2}] \cup [x_{2p+3}, \infty),$$

where $p \in \mathbb{N} \cup \{0\}$, $x_1 < x_2 < x_3$ and $x_j \leq x_{j+1}$ for each $j \geq 3$. Let us take a polynomial $F_1 \in \text{Pos}_{\geq 0}^2(K_2)$ and define the polynomial

$$F(x) := \prod_{y \in K_{01}} (x-y) \cdot F_1(x) \in \text{Pos}_{\geq 0}^2(K). \quad (12)$$

Let $S := \{g_1, \dots, g_s\}$ be the natural description of K . If F belongs to T_S^2 , then for every $e \in \{0, 1\}^s$ there exists $\sigma_e \in \sum M_n(\mathbb{C}[x])^2$ such that

$$F = \sum_{e \in \{0, 1\}^s} \sigma_e \underline{g}^e. \quad (13)$$

Since for every $y \in K_{01}$ and every $e \in \{0, 1\}^s$ we have $F(y) = 0$ and $\sigma_e \underline{g}^e(y) \succeq 0$, it follows from (13) that $\sigma_e \underline{g}^e(y) = 0$. Therefore $\prod_{y \in K_{01}} (x-y)$ divides each $\sigma_e \underline{g}^e$.

Claim. There exist $\tau_e \in \sum M_n(\mathbb{C}[x])^2$ and $h_e \in \text{Pos}_{\geq 0}^1(K_2)$ such that

$$\frac{\sigma_e \underline{g}^e}{\prod_{y \in K_{01}} (x - y)} = \tau_e h_e.$$

Proof of Claim. Let us take $y \in K_{01}$. We separate two possibilities.

1. $x - y$ divides σ_e : Then $\sigma_e \underline{g}^e = \hat{\sigma}_e \cdot (x - y)^2 \underline{g}^e$ where $\hat{\sigma}_e \in \sum M_n(\mathbb{C}[x])^2$ and $\frac{(x-y)^2 \underline{g}^e}{x-y} = (x - y) \underline{g}^e \in \text{Pos}_{\geq 0}^1(K_2)$.
2. $x - y$ does not divide σ_e : Then $x - y$ divides \underline{g}^e and hence $\sigma_e \underline{g}^e = \sigma_e \cdot (x - y) \hat{g}_e$ where $\hat{g}_e := \frac{\underline{g}^e}{x-y} \in \text{Pos}_{\geq 0}^1(K_2)$.

Repeating the above procedure for every $y \in K_{01}$ we obtain τ_e and h_e proving Claim.

Let S_2 be the natural description of K_2 . By [11, Theorem 2.2], $h_e \in T_{S_2}^1$. It follows that $F_1 = \sum_e \tau_e h_e \in T_{S_2}^2$.

We have proved that for $F_1 \in \text{Pos}_{\geq 0}^2(K_2)$ and $F \in \text{Pos}_{\geq 0}^2(K)$ defined by (12), from $F \in T_S^2$ it follows that $F_1 \in T_{S_2}^2$. Therefore, to find $F \in \text{Pos}_{\geq 0}^2(K)$ and $F \notin T_S^2$, it is sufficient to find $F_1 \in \text{Pos}_{\geq 0}^2(K_2)$ and $F_1 \notin T_{S_2}^2$. Let us define the set $K_3 := [x_1, x_2] \cup [x_3, \infty)$. By Claim 1 of Proposition 3.4, there exists a polynomial $F_1 \in \text{Pos}_{\geq 0}^2(K_3) \subseteq \text{Pos}_{\geq 0}^2(K_2)$ such that $F_1 \notin T_{S_2}^2$. This proves Case 1.

Case 2: K_1 is unbounded from below and bounded from above. Make a substitution $x \mapsto -x$ and observe that the set $-K_1$ is of the form in Case 1 and that the natural description of K maps into the natural description of $-K$.

Case 3: K_1 is unbounded from below and above. Let $d \in \mathbb{R}$ be the smallest endpoint of K_1 . Define the map $\lambda_d : \mathbb{R} \setminus \{d\} \rightarrow \mathbb{R}$ with $\lambda_d(x) := \frac{1}{d-x}$. Observe that $\lambda_d(K_1) =: K_2$ is the set of the form $[x_1, x_2] \cup [x_3, x_4] \cup \dots \cup [\hat{x}_{2m+1}, \infty)$ where $m \in \mathbb{N}$ and $x_j < x_{j+1}$ for every j . Let S_3 be the natural description of $\lambda_d(K)$. As in Case 1, construct the polynomial $F \in \text{Pos}_{\geq 0}^2(\lambda_d(K))$ such that $F \notin T_{S_3}^2$. Now $G(x) = x^{(2 \lceil \frac{\deg(F)}{2} \rceil)} \cdot F(d - \frac{1}{x}) \in \text{Pos}_{\geq 0}^2(K)$ and $G \notin T_S^2$. \square

Proof of Theorem 3.2.2 and 3.2.3. By Lemma 3.3, we may assume that S is the natural description of K . Let $d \in \mathbb{R}$ be an arbitrary point such that $d \notin K$. Define the map $\lambda_d : \mathbb{R} \setminus \{d\} \rightarrow \mathbb{R}$ with $\lambda_d(x) := \frac{1}{d-x}$. Observe that $\lambda_d(K)$ is the set of the form $[x_1, x_2] \cup \bigcup_{j=3}^m \{x_j\}$ where $m \geq 4$ and the points x_j are pairwise different. Further on, we may choose $d \in \mathbb{R}$ such that $x_1 < x_2 < x_3 < \dots < x_m$ or $x_m < x_{m-1} < \dots < x_3 < x_1 < x_2$. By substitution $x \mapsto -x$, we may assume that $x_1 < x_2 < x_3 < \dots < x_m$. Let $S_1 = \{g_1, \dots, g_s\}$ be the natural description of $\lambda_d(K)$. Notice that to prove the statement of the theorem, it is sufficient to find $F \in \text{Pos}_{\geq 0}^2(\lambda_d(K))$ of degree $2k$ such that $F \notin T_{S_1, 2k}^2$. By Claim 2 of Proposition 3.4, there is $F \in \text{Pos}_{\geq 0}^2(\lambda_d(K))$ of degree 2 such that $F \notin T_{S_1, 2}^2$. This concludes the proof. \square

Theorem 3.5 gives a characterization of the set $\text{Pos}_{\geq 0}^n(K)$ for unbounded sets K .

Theorem 3.5. *Suppose K is an unbounded closed semialgebraic set in \mathbb{R} and S the natural description of K . Then, for any $F \in \mathbb{H}_n(\mathbb{C}[x])$, the following are equivalent:*

1. $F \in \text{Pos}_{\geq 0}^n(K)$.
2. *There exists a polynomial $h \in \mathbb{R}[x]$ such that for every isolated point $w \in K$, $h(w) \neq 0$ and $h^2 F \in T_S^n$.*
3. *For every point $w \in \mathbb{C}$ there exists a polynomial $h \in \mathbb{R}[x]$ such that $h(w) \neq 0$ and $h^2 F \in T_S^n$.*

Proof. For the implication (3) \Rightarrow (2) construct h in the same way as in Remark 2.7 (replace the boundary of K with the set of its isolated points). The implication (2) \Rightarrow (1) is trivial. The proof of direction (1) \Rightarrow (3) is the same as the proof of Proposition 2.2, just that we use [11, Theorem 2.2] for the $n = 1$ case instead of [22, Theorem 5.17]. \square

4. Generalizations of the results to curves

In this section Theorem 2.1 is generalized to curves in \mathbb{R}^n . A characterization of sets S satisfying Theorem 4.1.1 was proved by Scheiderer in [21, Theorem 5.17] and [22, Corollary 4.4]. Using the same method as in the proof of Theorem 2.1 we obtain the implication 1. \Rightarrow 2. of the following theorem.

Theorem 4.1. *Suppose I is a prime ideal of $\mathbb{R}[\underline{x}]$ with $\dim(\frac{\mathbb{R}[\underline{x}]}{I}) = 1$ and let $\mathcal{Z}(I) := \{\underline{x} \in \mathbb{R}^d : f(\underline{x}) = 0 \text{ for every } f \in I\}$ be its vanishing set. Let $S := \{g_1, \dots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{x}]$ and $K_S = \{\underline{x} \in \mathbb{R}^d : g_1(\underline{x}) \geq 0, \dots, g_s(\underline{x}) \geq 0\}$ the associated semialgebraic set. Suppose the set $K_S \cap \mathcal{Z}(I)$ is compact. Then the following are equivalent:*

1. *The quadratic module $M_S^1 + I$ is saturated.*
2. *The n -th quadratic module $M_S^n + M_n(I)$ is saturated for every $n \in \mathbb{N}$.*

An example of a non-singular curve is the unit circle. Theorem 1.1 has an equivalent version for the unit complex circle \mathbb{T} (see [19] or [16]). By passing from complex numbers to pairs of real numbers and by Theorem 4.1, we obtain a generalization of this equivalent version to an arbitrary semialgebraic set in the unit circle. To explain this generalization we need some notation. Let us equip the set of $n \times n$ matrix Laurent polynomials $M_n(\mathbb{C}[z, \frac{1}{z}])$ with an involution $A(z)^* := \overline{A(\frac{1}{\bar{z}})}^T$. We denote by $\mathbb{H}_n(\mathbb{C}[z, \frac{1}{z}])$ the set of all $B \in M_n(\mathbb{C}[z, \frac{1}{z}])$ such that $B^* = B$, and by $\sum M_n(\mathbb{C}[z])^2$ the set of all finite sums of elements of the form $B^* B$ where $B \in M_n(\mathbb{C}[z])$. Let $\mathcal{S} = \{b_1, \dots, b_s\}$ be a finite set from $\mathbb{H}_1(\mathbb{C}[z, \frac{1}{z}])$ and $\mathcal{K}_{\mathcal{S}} = \{z \in \mathbb{T} : b_j(z) \geq 0, j = 1, \dots, s\}$ the associated semialgebraic set. Let the n -th matrix quadratic module generated by \mathcal{S} in $\mathbb{H}_n(\mathbb{C}[z, \frac{1}{z}])$ be

$$\mathcal{M}_{\mathcal{S}}^n := \{\tau_0 + \tau_1 b_1 + \dots + \tau_s b_s : \tau_j \in \sum M_n(\mathbb{C}[z])^2 \text{ for } j = 0, \dots, s\}.$$

We write $\text{Pos}_{\geq 0}^n(\mathcal{K}_{\mathcal{S}})$ for the set of elements from $\mathbb{H}_n(\mathbb{C}[z, \frac{1}{z}])$ which are positive semidefinite on $\mathcal{K}_{\mathcal{S}}$.

Corollary 4.2. $\mathcal{M}_{\mathcal{S}}^n = \text{Pos}_{\geq 0}^n(\mathcal{K}_{\mathcal{S}})$ iff \mathcal{S} satisfies the following conditions:

- (a) For every boundary point $a \in \mathcal{K}_{\mathcal{S}}$ which is not isolated there exists $k \in \{1, \dots, s\}$ such that $b_k(a) = 0$ and $\frac{db_k}{dz}(a) \neq 0$.
- (b) For every isolated point $a \in \mathcal{K}_{\mathcal{S}}$ there exist $k, l \in \{1, \dots, s\}$ such that $b_k(a) = b_l(a) = 0$, $\frac{db_k}{dz}(a) \neq 0$, $\frac{db_l}{dz}(a) \neq 0$ and $b_k b_l \leq 0$ on some neighborhood of a .

As an application of Corollary 4.2 we obtain the following improvement of Theorem 3.5:

Corollary 4.3. Suppose K is an unbounded closed semialgebraic set in \mathbb{R} and S the natural description of K . Then, for $F \in \mathbb{H}_n(\mathbb{C}[x])$, the following are equivalent:

- 1. $F \in \text{Pos}_{\geq 0}^n(K)$.
- 2. For every $w \in \mathbb{C} \setminus \mathbb{R}$ there exists $k_w \in \mathbb{N} \cup \{0\}$ such that

$$((x - \overline{w})(x - w))^{k_w} F \in M_S^n.$$

To prove Corollary 4.3 we need some preliminaries. Möbius transformations that map $\mathbb{R} \cup \{\infty\}$ bijectively into \mathbb{T} are exactly the maps of the form

$$\lambda_{z_0, w_0} : \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{T}, \quad \lambda_{z_0, w_0}(x) := z_0 \frac{x - w_0}{x - \overline{w_0}},$$

where $z_0 \in \mathbb{T}$ and $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Notice that $\lambda_{z_0, w_0}^{-1}(x) = \frac{z_0 \overline{w_0} - z_0 w_0}{x - z_0}$. If $F(x)$ is a matrix polynomial from $M_n(\mathbb{C}[x])$, then

$$\Lambda_{z_0, w_0, F}(z) := ((z - z_0)^*(z - z_0))^{\lceil \frac{\deg(F)}{2} \rceil} \cdot F(\lambda_{z_0, w_0}^{-1}(z))$$

is a matrix polynomial from $M_n(\mathbb{C}[z, \frac{1}{z}])$. Observe that

$$F(x) = \left(\frac{(x - \overline{w_0})(x - w_0)}{4 \cdot \Im(w_0)^2} \right)^{\lceil \frac{\deg(F)}{2} \rceil} \cdot \Lambda_{z_0, w_0, F}(\lambda_{z_0, w_0}(x)),$$

where $\Im(w_0)$ is the imaginary part of w_0 .

Proof of Corollary 4.3. The non-trivial direction is $1. \Rightarrow 2.$ Choose $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Observe that $\Lambda_{1, w_0, F}(z)$ belongs to the set $\text{Pos}_{\geq 0}^n(\mathcal{K}_{w_0})$ where $\mathcal{K}_{w_0} := \text{Cl}(\lambda_{1, w_0}(K))$ and $\text{Cl}(\cdot)$ is the closure operator. Let $S = \{g_1, \dots, g_s\}$ be the natural description of K . Then $\mathcal{S} := \{\Lambda_{1, w_0, g_1}(z), \dots, \Lambda_{1, w_0, g_s}(z)\}$ satisfies the conditions of Corollary 4.2 and hence $\Lambda_{1, w_0, F} \in \mathcal{M}_{\mathcal{S}}^n$. Therefore

$$\left(\frac{(x - \overline{w_0})(x - w_0)}{4 \cdot \Im(w_0)^2} \right)^{k_{w_0}} \cdot F(x) \in M_S^n,$$

where $k_{w_0} \in \mathbb{N} \cup \{0\}$ equals $k - \lceil \frac{\deg(F)}{2} \rceil$ with k being the degree of the summand of the highest degree in the expression of $\Lambda_{1, w_0, F}(z)$ as the element of $\mathcal{M}_{\mathcal{S}}^n$. \square

Remark 4.4. By a similar but more technical proof we can show, that Corollary 4.3.2 is true for all $w \in \mathbb{C} \setminus K$, i.e., it is true also for $w \in \mathbb{R} \setminus K$.

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